MTH 1420, SPRING 2012
DR. GRAHAM-SQUIRE

LAB 8: ESTIMATING THE SUM OF A SERIES

Names: ______________________________________

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1. Instructions

(1) If you choose to work in a group, your group should write up and turn in one completed lab at the start of the next lab period. You can use this sheet as a cover sheet for the lab you turn in. **Each member of the group should write up at least part of the lab**, but you should check each other's work since everyone in the group gets the same score.

2. Introduction

We have recently been talking about how to find out if a series converges or diverges. If a series converges, though, unless the series is a geometric or telescoping series we have no way of finding out exactly what it converges *to*. In these cases, it is often useful to be able to estimate what the sum of the series is.

3. Estimating using partial sums

The easiest way to get an estimate for the sum of a series is simply to add up a certain number of terms. In other words, you can find an estimate for a series just by finding a partial sum.

Exercise 1. Estimate the sum of following p-series by adding up the first ten terms of the partial sum (that is, find s_{10}):

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^5}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{200}{n^2}$

Exercise 2. For the two series above, how close do you think the estimate s_{10} is to the actual sum of the series?

The problem with this method of estimation is that you have no idea how close your approximation is to the actual value of the series, and you have no idea how far you have to add up the partial sum in order to make it close to the actual sum of the series. It would help to have an idea of how much error you incur by adding up only the first ten terms as opposed to adding up, say, the first 100 terms.

4. Remainders, error, and the Integral Test

Remainders:

Definition 3. Let $\sum_{n=1}^{\infty} a_n$ be a convergent series with sum s. That is, $\sum_{n=1}^{\infty} a_n = s$. We can break the series into two parts and write

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{k} a_n + \sum_{n=k+1}^{\infty} a_n. \tag{1}$$

The first part is what we call the partial sum $s_k = \sum_{n=1}^{k} a_n$, and the second part we will call the remainder at k: $R_k = \sum_{n=k+1}^{\infty} a_n$.

Exercise 4. Write an equation for R_k in terms of s and s_k (This follows directly from equation (1) and the definitions of s, s_k and R_k).

The sum of a series, graphically:

We will now explore how to think of the sum of a series (and the remainder defined above) graphically, and compare with what we know about integration and area under curves.

Exercise 5. The Harmonic series, graphically: The goal of this exercise is to give a graphical representation of the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

and to compare it the improper integral $\displaystyle\int_1^{\infty} \frac{1}{x}\,dx$.

(a) Using a full sheet of paper, sketch the graph of $y = 1/x$ in the first quadrant, from $x = 0$ to $x = 10$, $y = -1$ to $y = 2$. Your graph needs to be big because we will be filling the space with other things. Notice that the area under the curve from $x = 1$ to the right is the same thing as the improper integral $\int_1^{\infty} \frac{1}{x}\,dx$.

(b) Draw vertical lines between the graph and the x-axis at every integer 1, 2, 3, . . . , 9.

(c) At $x = 1$, draw a horizontal line to the left, from the graph to the y-axis. At $x = 2$, draw a horizontal line from the graph to the vertical line at $x = 1$. At $x = 3$, draw a horizontal line from the graph to the vertical line at $x = 2$. Repeat for all the other integers, making rectangles that look like a Riemann sum using the right endpoint for evaluation.

(d) Write an improper integral to represent the area under the curve from $x = 1$ to infinity.

(e) Calculate the areas of the rectangles (starting at $x = 1$ and moving to the right) and write a Riemann sum to represent the sum of the areas of all the rectangles to the right of x=1. (Your answer should look like the harmonic series but with one small difference).

(f) Compare your answers from questions (d) and (e). Namely, which one is bigger, and why? (You can explain just by referencing what the picture tells you).

(g) Repeat questions (a) through (f), with one small change. When you draw the horizontal lines in part (c), you should now draw them to the right for one unit, then draw down to connect with the vertical line below. This will effectively create rectangles whose tops now lie *above* the curve. Question (d) will have the same answer, but (e) and (f) should be different.

Conclusions: From the exercise above, you should be able to conclude the following:

$$\sum_{n=1}^{\infty} \frac{1}{n} > \int_1^{\infty} \frac{1}{x}\, dx > \sum_{n=2}^{\infty} \frac{1}{n}.$$

This is one way to justify that the harmonic series diverges, because we just showed that the harmonic series will sum to something greater than the divergent improper integral $\int_1^{\infty} \frac{1}{x}\, dx$. The exercise has another use, though. In particular, you can use the graphs you made and similar reasoning to show that:

$$\int_1^{\infty} \frac{1}{x}\, dx > \sum_{n=2}^{\infty} \frac{1}{n} > \int_2^{\infty} \frac{1}{x}\, dx$$

For the harmonic series this is not very useful because each of the expressions are infinity. However, you can use the same kind of reasoning on *any* series with positive terms to conclude the Remainder Estimate for the series $\sum_{n=1}^{\infty} a_n$

$$\int_{k+1}^{\infty} f(x)\, dx \le R_k \le \int_k^{\infty} f(x)\, dx$$

where $f(x)$ is a continuous decreasing function with $f(n) = a_n$ for all integers n. This inequality should make sense to you after doing the exercise above, **if it does not make sure to ask Dr. G-S so he can explain it to you**. Since R_k is the difference between the actual sum of a series and the partial sum s_k, we can think of R_k as the error of using s_k as an approximation for the sum of a series.

5. Uses of the Remainder Estimate

The Remainder estimate for the Integral Test is not the best because it only works if the series you are looking at is integrable, and many are not. With that said, it is useful for two purposes when you do have an integrable function: the remainder estimate gives an estimate for how close your approximation is to the actual sum of a series, and it can tell you how many terms you need to add up in order to get an estimate that is within a certain amount of accuracy. We will do examples of both.

Exercise 6. Estimate the error in using s_{10} as an approximation for (i) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^5}$ and (ii) $\displaystyle\sum_{n=1}^{\infty} \frac{200}{n^2}$. The idea is to get an approximation for the value of R_{10} using the remainder estimate for the integral test. Your estimate for R_{10} will have a lower bound and an upper bound that you find from evaluating $\displaystyle\int_{10}^{\infty}$ and $\displaystyle\int_{11}^{\infty}$. Note that this gives you an answer to something you guessed at earlier in the lab.

Exercise 7. Find a value of k such that s_k is within 0.00001 of the sum of the series $\displaystyle\sum_{n=1}^{\infty} \frac{200}{n^2}$. This will tell you how far you have to add up partial sums in order to get within 0.00001 of the actual sum for the series. To do this, you need to find k such that $R_k \leq 0.00001$, which means you only need to use half of the remainder estimate. Namely, you are just trying to find a value of k such that

$$\int_{k}^{\infty} \frac{200}{x^2}\, dx \leq 0.0001$$

which you can do by integrating the left hand side, then solving the inequality for k.

Exercise 8. How many terms of the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ would you need to add to find its sum to within 0.01?